

# GLOBAL EXISTENCE FOR SEMILINEAR REACTION-DIFFUSION SYSTEMS ON EVOLVING DOMAINS

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**ABSTRACT.** We present global existence results for solutions of reaction-diffusion systems on evolving domains. Global existence results for a class of reaction-diffusion systems on fixed domains are extended to the same systems posed on spatially linear isotropically evolving domains. The results hold without any assumptions on the sign of the growth rate. The analysis is valid for many systems that commonly arise in the theory of pattern formation. We present numerical results illustrating our theoretical findings.

## 1. INTRODUCTION

Since their seminal introduction by Turing [1952], reaction-diffusion systems (RDS's) have constituted a standard framework for the mathematical modelling of pattern formation in chemistry and biology. Recent advances in mathematical modelling and developmental biology identify the important role of *domain evolution* as central in the formation of patterns, both empirically [Kondo and Asai, 1995] and computationally [Comanici and Golubitsky, 2008; Crampin et al., 1999; Madzvamuse and Maini, 2007]. In this respect, many numerical studies, such as Madzvamuse [2006] and Barrass et al. [2006], of RDS's on evolving domains are available. Yet, fundamental mathematical questions such as existence and regularity of solutions of RDS's on evolving domains remains an important open question [Kelkel and Surulescu, 2009].

Numerous studies on the stability of solutions of RDS's on fixed domains are available, for example, Hollis et al. [1987]; Rothe [1984]; Wei and Winter [2008], but very little literature regarding the stability of solutions of RDS's on evolving domains. Madzvamuse et al. [2010] provides a linear stability analysis of RDS's on continuously evolving domains, and Labadie [2008] examines the stability of solutions of RDS's on monotonically growing surfaces. Our discussion here differs from all these studies in that we focus on planar evolving domains and we show existence, uniqueness and stability, for an entire class of RDS's on evolving domains independently of the rate of evolution. In this article we prove the stability of solutions of RDS's on a particular, but fundamentally important, class of time-evolving domains: that of *bounded spatially linear isotropically evolving domains*.

We show that if a RDS fulfils a restricted version of certain *stability conditions*, introduced by Morgan [1989] for fixed domains, then the RDS fulfils the same stability conditions on any bounded spatially linear isotropic evolution of the domain. We thus prove that, under certain conditions, the existence and uniqueness for a RDS on a fixed domain implies the existence and uniqueness for the corresponding RDS on an evolving domain. This is, to our best knowledge, the first result that holds independently of the growth rate and is thus valid on growing or contracting domains as well as domains that exhibit periods of growth and periods of contraction. Our analysis rigourously justifies computations for this type of domain evolution, and we

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illustrate our results with benchmark computations using a moving finite element (Lagrangian) approach.

The outline of our discussion goes as follows: In §2 we state our model problem together with the form of domain evolution that we consider, and present a transformation of our model system to the Lagrangian framework that helps in proving global existence of solutions. In §3 we review the existence results for RDS's on fixed domains which will form the basis of our analysis. In §4 we state and prove the central results of this work; in particular, we extend the existence results cited in §3 to problems posed on evolving domains. In §5 we illustrate some specific applications of our results, in particular those of significance in the field of biological pattern formation. We focus on growth functions commonly encountered in the field of developmental biology for which our analysis is valid and show the applicability of our analysis to some of the important reaction kinetics encountered in the theory of biological pattern formation. In §6 we present numerical results for a RDS posed on a periodically evolving domain. We present a moving finite element scheme and a fixed domain finite element scheme to approximate the solution of a RDS posed on the evolving and the Lagrangian frame respectively. In §7 we summarise our findings and indicate future research directions.

## 2. RDS'S ON CONTINUOUSLY EVOLVING DOMAINS

Let  $\mathbf{u}(\mathbf{x}, t)$  be a  $(m \times 1)$  vector of concentrations of chemical species, with  $\mathbf{x} \in \mathbb{R}^n$ , the time-dependent spatial variable and  $t \in [0, T]$ ,  $T > 0$ , the time variable. The model problem we wish to consider is a semilinear RDS posed on a continuously evolving domain (see Madzvamuse [2000] for details of the derivation), given by,

$$(2.1) \quad \partial_t \mathbf{u}(\mathbf{x}, t) + [\nabla \cdot (\mathbf{a} : \mathbf{u})](\mathbf{x}, t) = \mathbf{f}(\mathbf{u}(\mathbf{x}, t)) + \mathbf{D} \Delta \mathbf{u}(\mathbf{x}, t) \quad \text{for } \mathbf{x} \in \Omega_t \text{ and } t \in (0, T],$$

where  $\Omega_t \subset \mathbb{R}^n$  ( $n < \infty$ ) is a  $C^{2+\gamma}(\Omega)$ , simply connected, bounded and continuously deforming domain with respect to  $t$ . The function  $\mathbf{f}$  is a  $(m \times 1)$  vector of nonlinear coupling terms that is locally Lipschitz,  $\mathbf{D}$  is a diagonal matrix with strictly positive entries on the diagonal,  $\mathbf{a}$  is a flow velocity generated by the evolution of the domain and  $(\mathbf{a} : \mathbf{u}) := (\mathbf{a}u_1, \dots, \mathbf{a}u_m)^T$  by definition.

**2.1. Assumption** (Flow velocity) We assume that the flow velocity  $a_i(\mathbf{x}, t)$  is identical to the domain velocity, i.e.,

$$a_i = \partial_t x_i \quad i = 1, \dots, n,$$

as is standard in the derivation of RDS's on evolving domains on application of Reynold's Transport Theorem [Acheson, 1990].

To simplify the exposition we take boundary conditions to be of homogenous Neumann type. Morgan [1989] considers a much wider class of boundary conditions and our analysis may be extended to this more general setting. We are primarily interested in patterns that arise as a result of self-organisation prompting the consideration of homogenous Neumann boundary conditions. We take the initial condition for each  $u_i$  to be bounded and nonnegative. Our model problem thus takes the form (2.1), equipped with the following boundary and initial conditions:

$$(2.2) \quad \begin{cases} [\vec{\nu} \cdot \nabla \mathbf{u}](\mathbf{x}, t) = 0, & \mathbf{x} \in \partial\Omega_t, t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega_0. \end{cases}$$

**2.2. Lagrangian transformation.** For our analysis it is more convenient to work with problems defined on a time-independent domain. Therefore, we introduce a transformation that maps our model problem (2.1) from a time-dependent domain to a fixed domain (see Baines [1994] for a more detailed discussion of this approach). In order to do this, without too many technical complications, we will restrict our attention to special evolutions of the domain, described next.

**2.3. Assumption** (Isotropic domain evolution) We assume the domain  $\Omega_t$  to evolve by obeying a bounded spatially linear isotropic domain deformation, i.e.,

$$(2.3) \quad \mathbf{x} = \rho(t)\boldsymbol{\xi} \quad \text{for all } (\boldsymbol{\xi}, t) \in \Omega_0 \times [0, T] \text{ and all } \mathbf{x} \in \Omega_t,$$

with  $\rho \in C^2([0, T]; (0, \infty))$  and where  $\boldsymbol{\xi}$  represents the spatial coordinates of the initial domain. This assumption and Assumption 2.1 imply that

$$(2.4) \quad \mathbf{a}(\mathbf{x}, t) = \dot{\rho}(t)\boldsymbol{\xi},$$

where  $\dot{\rho} := \frac{d\rho}{dt}$ .

Hence, we obtain the following transformed problem on the initial domain (see Madzvamuse and Maini [2007] for details), with

$$(2.5) \quad \hat{\mathbf{u}}(\boldsymbol{\xi}, t) = \mathbf{u}(\rho(t)\boldsymbol{\xi}, t) \quad \text{for } t \in [0, T] \text{ and } \boldsymbol{\xi} \in \Omega_0,$$

we have

$$(2.6) \quad \begin{cases} \partial_t \hat{\mathbf{u}} + n \frac{\dot{\rho}}{\rho} \hat{\mathbf{u}} = \mathbf{f}(\hat{\mathbf{u}}) + \frac{D}{\rho^2} \Delta \hat{\mathbf{u}} & \text{on } \Omega_0 \times (0, T], \\ [\vec{\nu} \cdot \nabla \hat{\mathbf{u}}](\boldsymbol{\xi}, t) = 0 & \text{on } \partial\Omega_0, t > 0, \\ \hat{\mathbf{u}}(\boldsymbol{\xi}, 0) = \hat{\mathbf{u}}_0(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \Omega_0, \\ 0 \leq \hat{\mathbf{u}}_0(\boldsymbol{\xi}) < \infty, & \end{cases}$$

where  $\Omega_0$  is the initial spatial domain,  $n$  is the spatial dimension and the Laplacian is now taken with respect to  $\boldsymbol{\xi}$ . The local and global existence results that we utilise from the existing literature require the coefficients on our diffusion term to be independent of time; to this end we introduce the following proposition:

**2.4. Proposition** (Time rescaling [Labadie, 2008]) Let  $\mathbf{u}$  be a solution of (2.1), rescaling time via the change of variables

$$(2.7) \quad s(t) := \int_0^t \frac{dr}{\rho(r)^2},$$

and denoting  $S := s(T)$ . We have,  $\mathbf{u}(\rho(t)\boldsymbol{\xi}, t) = \tilde{\mathbf{u}}(\boldsymbol{\xi}, s)$ , where  $\tilde{\mathbf{u}}$  satisfies

$$(2.8) \quad \begin{cases} \partial_s \tilde{\mathbf{u}} + n \rho \dot{\rho} \tilde{\mathbf{u}} = \rho^2 \mathbf{f}(\tilde{\mathbf{u}}) + D \Delta \tilde{\mathbf{u}} & \text{on } \Omega_0 \times (0, S], \\ [\vec{\nu} \cdot \nabla \tilde{\mathbf{u}}](\boldsymbol{\xi}, s) = 0 & \boldsymbol{\xi} \in \partial\Omega, s > 0, \\ \tilde{\mathbf{u}}(\boldsymbol{\xi}, 0) = \tilde{\mathbf{u}}_0(\boldsymbol{\xi}) & \boldsymbol{\xi} \in \Omega. \end{cases}$$

Furthermore, if  $\mathbf{f}(\mathbf{u})$  is locally Lipschitz in  $\mathbf{u}$  then  $\tilde{\mathbf{f}}(\tilde{\mathbf{u}}(\vec{\xi}, s), s) = \rho^2(s) \mathbf{f}(\tilde{\mathbf{u}}(\vec{\xi}, s)) - n \rho(s) \dot{\rho}(s) \tilde{\mathbf{u}}(\vec{\xi}, s)$ , is locally Lipschitz in  $\tilde{\mathbf{u}}$ .

**Proof** We note that with domain evolution of the form considered in this study, there exist  $C_1, C_2$  such that  $0 < \rho \leq C_1 < \infty$  and  $\|\dot{\rho}\|_{L^\infty[0, T]} \leq C_2 < \infty$ . Applying the rescaling (2.7), we see that for any function  $g \in C^1[0, T]$

$$(2.9) \quad \partial_s g(t) = \partial_s t(s) \partial_t g(t) = \rho^2(t) \partial_t g(t).$$

Defining  $\tilde{\mathbf{u}}(\boldsymbol{\xi}, s) := \mathbf{u}(\rho(t)\boldsymbol{\xi}, t)$  and multiplying problem (2.6) by  $\rho^2$ , we obtain problem (2.8). Clearly since  $\mathbf{f}(\mathbf{u})$  is locally Lipschitz in  $\mathbf{u}$ , Assumption 2.3 implies  $\mathbf{f}(\tilde{\mathbf{u}}, s)$  is locally Lipschitz in  $\tilde{\mathbf{u}}$  and globally Lipschitz in  $s$ .  $\square$

### 3. BASIC THEORETICAL SETTING FOR RDS'S ON FIXED DOMAINS

We now summarise the existing results for RDS's on fixed domains which will form the basis of our analysis on evolving domains. The following result is a straightforward generalisation of Hollis et al. [1987, Prop. 1].

**3.1. Theorem** (Local existence) Let Assumption 2.3 hold. Problem (2.8) admits a unique local solution. Furthermore, defining the unique maximal solution of (2.8) by

$$(3.1) \quad \tilde{\mathbf{u}} : \Omega_0 \times [0, T_{\max}) \rightarrow \mathbb{R}^m,$$

there exists a function  $\mathbf{N} \in C([0, T_{\max}); \mathbb{R}^m)$  such that,

$$(3.2) \quad \|\tilde{u}_i(\cdot, s)\|_{L_\infty(\Omega_0)} \leq N_i(s) \quad \text{for } i \in [1, \dots, m] \text{ and } s \in [0, T_{\max}).$$

Finally, if  $T_{\max} < \infty$ ,

$$(3.3) \quad \lim_{s \rightarrow T_{\max}} \left( \sum_{i=1}^m \|\tilde{u}_i(\cdot, s)\|_{L_\infty(\Omega_0)} \right) = \infty.$$

**3.2. Global existence on fixed domains.** If  $\rho(t) = 1$  for all  $t \in [0, T]$ , Problem (2.8) becomes, find  $\tilde{\mathbf{u}} : \Omega_0 \times (0, T] \rightarrow \mathbb{R}^m$  such that,

$$(3.4) \quad \begin{cases} \partial_t \tilde{\mathbf{u}} = \mathbf{f}(\tilde{\mathbf{u}}) + \mathbf{D} \Delta \tilde{\mathbf{u}}, & \text{on } \Omega_0 \times (0, T], \\ [\vec{\nu} \cdot \nabla \tilde{\mathbf{u}}](\boldsymbol{\xi}, t) = 0, & \boldsymbol{\xi} \in \partial\Omega_0, t > 0, \\ \tilde{\mathbf{u}}(\boldsymbol{\xi}, 0) = \tilde{\mathbf{u}}_0(\boldsymbol{\xi}), & \boldsymbol{\xi} \in \Omega_0, \end{cases}$$

where we have used the fact that  $s(t) = t$  (cf. (2.7)).

**3.3. Definition** (Invariant region)  $\Sigma \subset \mathbb{R}^m$  is called an invariant region for the solution of the reaction-diffusion system (3.4) if for any solution  $\mathbf{u}$ ,

$$(3.5) \quad \mathbf{u}(\boldsymbol{\xi}, 0) \in \Sigma \implies \mathbf{u}(\boldsymbol{\xi}, t) \in \Sigma \quad \text{for all } t \in (0, T].$$

**3.4. Assumption** (Positive solutions) We assume hereon that

$$(3.6) \quad f_i(\mathbf{u})|_{u_i=0} \geq 0 \text{ for all } t \in [0, T],$$

and for  $\mathbf{f} \notin C^1(\mathbb{R}_+^m; \mathbb{R}^m)$ , the strict inequality

$$(3.7) \quad f_i(\mathbf{u})|_{u_i=0} > 0 \text{ for all } t \in [0, T].$$

Assumption 3.4 together with the positivity of our initial data, implies  $\mathbb{R}_+^m$  which we refer to as the positive quadrant, is an invariant region for the solutions of problem (3.4) (see Smoller [1994, Th.14.7, 14.11 pp.200–203]).

**3.5. Remark** (General invariant regions) Assumption 3.4 may be relaxed. The proof of our existence results only requires bounded initial data and the existence of an invariant region. Furthermore, consideration of the positive quadrant alone is sufficient for our studies.

**3.6. Lyapunov stability conditions.** We now introduce a Lyapunov function for the dynamical system defined by (3.4) when the initial condition  $\tilde{\mathbf{u}}_0$  varies which is used to prove global existence and a restricted version of the conditions it is required to fulfil [Morgan, 1989].

Suppose  $\mathbf{f}$  is as defined in problem (3.4) and that there exists a function  $H \in C^2(\mathbb{R}_+; \mathbb{R})$  and  $h_i \in C^2(\mathbb{R}_+; \mathbb{R})$  for each  $i = 1, \dots, m$ , such that

$$(3.8) \quad H(\mathbf{z}) = \sum_{i=1}^m h_i(z_i) \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m,$$

$$(3.9) \quad h_i(z_i), h_i''(z_i) \geq 0 \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m,$$

$$(3.10) \quad H(\mathbf{z}) \rightarrow \infty \iff \mathbf{z} \rightarrow \infty \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m.$$

Suppose there exists  $\mathbf{A} = (a_{ij}) \in (\mathbb{R})^{m \times m}$  satisfying  $a_{ij} \geq 0, a_{ii} > 0$  with  $1 \leq i, j \leq m$  such that for some  $r, k_1, k_2 \in \mathbb{R}_+$  independent of  $j$ , we have

$$(3.11) \quad \sum_{i=1}^j a_{ij} h_i'(z_i) f_i(\mathbf{z}) \leq k_1 (H(\mathbf{z}))^r + k_2 \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m, j \leq m.$$

Suppose there exist  $q, k_3, k_4 \in \mathbb{R}_+$  such that for  $1 \leq i \leq m$ , we have

$$(3.12) \quad h_i'(z_i) f_i(\mathbf{z}) \leq k_3 (H(\mathbf{z}))^q + k_4, \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m.$$

Suppose there exist  $k_5, k_6 \geq 0$  such that

$$(3.13) \quad \nabla H(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z}) \leq k_5 H(\mathbf{z}) + k_6 \quad \text{for all } \mathbf{z} \in \mathbb{R}_+^m.$$

**3.7. Theorem** (A priori estimates [Morgan, 1989]) Let conditions (3.8), (3.9) and (3.13) hold and let  $\tilde{\mathbf{u}}$  be a solution of problem (3.4). The following a priori estimates hold,

$$(3.14) \quad \left\| \int_{\tau}^t H(\tilde{\mathbf{u}}(\cdot, s)) \, ds \right\|_{L^\infty(\Omega_0)} \leq g(t) \quad \text{for } 0 \leq \tau < t < T_{\max},$$

$$(3.15) \quad \int_0^t \int_{\Omega_0} H(\tilde{\mathbf{u}}(\xi, s))^2 \, d\xi \, ds \leq \tilde{g}(t) \quad \text{for } 0 \leq t < T_{\max},$$

where  $g, \tilde{g} \in C[0, \infty)$ .

**3.8. Theorem** (Global existence on fixed domains [Morgan, 1989]) If conditions (3.8)—(3.12) hold, with  $r$  from condition (3.11) satisfying  $r < (1+a)$ ,  $a \in \mathbb{R}_+$ ,  $\tilde{\mathbf{u}}$  is a solution of problem (3.4) and if there exists  $g \in C[0, \infty)$  such that

$$(3.16) \quad \left\| \int_{\tau}^t \left| H(\tilde{\mathbf{u}}(\cdot, s)) \right|^a \, ds \right\|_{L^\infty(\Omega_0)} \leq g(t) \quad \text{for } 0 \leq \tau < t < T_{\max},$$

then  $T_{\max} = \infty$ . Alternatively, if conditions (3.8)—(3.12) hold, with  $r$  from condition (3.11) satisfying  $r < (1 + \frac{2b}{n+2})$ ,  $b > 0$  and where  $n$  represents the spatial dimension,  $\tilde{\mathbf{u}}$  solves a problem of the form (3.4) and if there exists  $\tilde{g} \in C[0, \infty)$  such that

$$(3.17) \quad \int_0^t \int_{\Omega_0} \left| H(\tilde{\mathbf{u}}(\xi, s)) \right|^b \, d\xi \, ds \leq \tilde{g}(t) \quad \text{for } 0 \leq t < T_{\max},$$

then  $T_{\max} = \infty$ .

Specifically if  $r$  from condition (3.11) satisfies  $r < 2$  or if  $\Omega_0 \subset \mathbb{R}$  with  $r < \frac{7}{3}$  and the remaining conditions (3.8)—(3.13) are satisfied then  $T_{\max} = \infty$ .

## 4. GLOBAL EXISTENCE ON EVOLVING DOMAINS

In this section we show that, if the stability conditions in §3.6 are valid for Problem (3.4) then they remain valid under any evolution of the domain fulfilling Assumption 2.3, given a suitable assumption on the structure of  $H$ . We also extend the previous a priori estimates and existence results of Morgan [1989] to problems with time dependent  $\mathbf{f}$ .

**4.1. Assumption** (Polynomial Lyapunov function) We assume the Lyapunov function introduced in §3.6 is of the following form

$$(4.1) \quad H(\mathbf{z}) = \sum_{i=1}^m z_i^{p_i}, \quad p_i \geq 1 \text{ for } i = 1, \dots, m.$$

**4.2. Remark** (Polynomial growth restriction) Assumption 4.1 is somewhat natural. Condition (3.12) is essentially a polynomial type growth restriction on the zero order terms [Morgan, 1989]. Assumption 4.1 can be viewed as the explicit analogue of the polynomial growth restriction on the zero order terms implicit in (3.12).

**4.3. Lemma** (Equivalence of Lyapunov functions) Suppose Assumptions 2.3, 3.4 and 4.1 hold. Let the Lyapunov stability conditions in §3.6 be satisfied by  $H$  and  $\mathbf{f}$ . Then the conditions in §3.6 with  $r$  (cf. (3.11)) replaced by  $\tilde{r} := \max(1, r)$ , are satisfied by  $H$  and  $\tilde{\mathbf{f}}$  (cf. (4.2)) in place of  $\mathbf{f}$ .

**Proof** We denote the zero order term in Problem (2.8) by

$$(4.2) \quad \tilde{\mathbf{f}}(\tilde{\mathbf{u}}(\vec{\xi}, s), s) := \rho^2(s) \mathbf{f}(\tilde{\mathbf{u}}(\vec{\xi}, s)) - n\dot{\rho}(s)\rho(s)\tilde{\mathbf{u}}(\vec{\xi}, s).$$

The positive quadrant remains an invariant region for the solutions of our evolving domain problem since

$$(4.3) \quad \tilde{f}_i(\tilde{\mathbf{u}}(\vec{\xi}, s), s)|_{u_i=0} = \rho^2(s) f_i(\tilde{\mathbf{u}}(\vec{\xi}, s))|_{u_i=0},$$

thus Assumption 3.4 implies  $\mathbb{R}_+^m$  is an invariant region for the solutions of problem (2.8). Let  $k_i, i = 1, \dots, 6, q, r$  and  $\mathbf{A}$  be as defined in §3.6, for which conditions (3.8)—(3.13) hold for problem (3.4). Denote  $C_1 := \|\rho\|_{L_\infty[0,T]}$  and  $C_2 := \|\dot{\rho}\|_{L_\infty[0,T]}$ ; these are well defined real numbers thanks to Assumption 2.3. We now show that conditions (3.8)—(3.13) hold with the same  $H, \mathbf{f}$  replaced by  $\tilde{\mathbf{f}}$  and  $r$  from (3.11) replaced by  $\tilde{r}$ , where  $\tilde{r} = \max(1, r)$ .

Clearly conditions (3.8)—(3.10) are still satisfied as they depend only on  $H$  which is unchanged. Condition (3.11) holds since

$$(4.4) \quad \sum_{i=1}^j a_{ij} h'_i \tilde{f}_i = \sum_{i=1}^j a_{ij} h'_i (\rho^2 f_i - n\dot{\rho}\rho \tilde{u}_i) \leq (k_1(H)^r + k_2)C_1^2 + nC_1C_2 \sum_{i=1}^j a_{ij} h'_i \tilde{u}_i,$$

by the stability of the fixed domain problem. Assumption 4.1 gives,

$$(4.5) \quad \begin{aligned} \sum_{i=1}^j a_{ij} h'_i \tilde{f}_i &\leq (k_1(H)^r + k_2)C_1^2 + nC_1C_2 \sum_{i=1}^j a_{ij} p_i h_i \\ &\leq (k_1(H)^r + k_2)C_1^2 + k_7 H \leq (k_1C_1^2 + k_7)(H)^{\tilde{r}} + k_8, \end{aligned}$$

where

$$\tilde{r} = \max(1, r) \text{ and } k_8 := \begin{cases} k_2C_1^2 + k_1C_1^2, & \text{if } r < 1, \\ k_2C_1^2, & \text{if } r = 1, \\ k_2C_1^2 + k_7, & \text{if } r > 1. \end{cases}$$

Condition (3.12) holds since

$$\begin{aligned}
 h'_i \tilde{f}_i &\leq C_1^2 h'_i f_i + n C_1 C_2 p_i h_i \\
 (4.6) \quad &\leq k_3 C_1^2 (H)^q + k_4 C_1^2 + n C_1 C_2 \max_i(p_i) H \\
 &\leq k_{10} H^{\tilde{q}} + k_{11},
 \end{aligned}$$

where

$$\tilde{q} = \max(1, q) \text{ and } k_{11} := \begin{cases} k_4 C_1^2 + k_3 C_1^2, & \text{if } q \leq 1, \\ k_4 C_1^2, & \text{if } q = 1, \\ k_4 C_1^2 + k_9, & \text{if } q > 1. \end{cases}$$

Condition (3.13) holds since

$$(4.7) \quad \nabla H \cdot \tilde{\mathbf{f}} \leq \sum_{i=1}^m C_1^2 h'_i f_i + n C_1 C_2 p_i h_i \leq (k_5 H + k_6) C_1^2 + k_{12} H.$$

Thus the positive quadrant remains an invariant region for the solutions of problem (2.8) and the Lyapunov stability conditions in §3.6 are satisfied, completing the proof.  $\square$

**4.4. Remark** (Applicability of Morgan [1989] to systems with time dependent zero order terms) Suppose the reaction function  $\tilde{\mathbf{f}}(\tilde{\mathbf{u}}(\boldsymbol{\xi}, t), t)$  is locally Lipschitz with respect to  $\tilde{\mathbf{u}}$  and  $t$ , and suppose that the Lyapunov function  $H$  depends only on  $\tilde{\mathbf{u}}$ . Then, Theorems 3.7 and 3.8 remain applicable [Morgan, 1989, (5.5)], [Morgan and Hollis, 1995, Th. 1.1] and [Bendahmane and Saad, 2010, Th. 4]. Thus, the Lipschitz result of Proposition 2.4, the structural Assumption 4.1 and the equivalence of Lyapunov functions proved in Lemma 4.3 imply that Theorems 3.7 and 3.8 are applicable for solutions of (2.8).

For completeness, we include a proof of Theorems 3.7 and 3.8 for solutions of Problem (2.8), in Appendices A and B respectively. To remain concise we prove a sufficient existence result for the examples presented in §5 and briefly sketch the full proof of Theorems 3.7 and 3.8.

The results of Morgan and Hollis [1995] apply for systems with time dependent diffusion. This may allow treatment of more general domain evolution where the rescaling carried out in §2 yields a system with time dependent diffusion. We leave this generalisation for future studies.

**4.5. Theorem** (Global existence of solutions on evolving domains) Let Assumptions 2.3, 3.4 and 4.1 hold and suppose  $H$ ,  $\mathbf{f}$  and  $r$  satisfy the conditions in §3.6 with  $r < 2$  or if  $\Omega \subset \mathbb{R}$ ,  $r < \frac{7}{3}$  (cf. (3.11)). Then, Problem (2.1) admits a global classical solution.

**Proof** Application of the results in §2.2 allows us to show existence for the transformed Problem (2.8) defined on a fixed domain. Theorem 3.1 gives the existence of a unique non-continuable classical solution. From Lemma 4.1 the stability conditions in §3.6 hold with  $\tilde{\mathbf{f}}$  (cf. (4.2)),  $H$  and  $r < 2$  or if  $\Omega \subset \mathbb{R}$ ,  $r < \frac{7}{3}$ . Theorem 3.7 gives an a priori estimate for  $H$ . Theorem 3.8 implies  $T_{\max} = \infty$  (cf. (3.1)) completing the proof.  $\square$

## 5. APPLICATIONS

In this section we illustrate some applications of Theorem 4.5. We present different forms of admissible domain evolution that fulfil Assumption 2.3. We show that Assumption 4.1 is applicable to some commonly encountered models in chemistry and biology. We concentrate on RDS's which admit Turing instabilities, as the main focus of our research is biological pattern formation.

We identify and describe Lyapunov functions and constants that imply global existence for the fixed domain problem and thus for the evolving domain problem by Theorem 4.5.

**5.1. Admissible domain evolution.** We now provide some commonly encountered examples of domain evolution in developmental biology for which Assumption 2.3 holds:

- Logistic evolution on any finite positive time interval

$$(5.1) \quad \rho(t) = \frac{e^{r_g t}}{1 + \frac{1}{K}(e^{r_g t} - 1)}, \quad t \in [0, T],$$

where  $r_g \geq 0$  is the growth rate and  $K > 1$  is the carrying capacity (limiting size of the evolving domain).

- Exponential evolution on any finite positive time interval

$$(5.2) \quad \rho(t) = e^{r_g t}, \quad t \in [0, T].$$

- Linear evolution on any finite positive time interval

$$(5.3) \quad \rho(t) = 1 + r_g t, \quad t \in [0, T],$$

where  $r_g > -\frac{1}{T}$ .

**5.2. Admissible kinetics.** We now present some of the commonly encountered reaction kinetics of problem (2.1) for which the analysis of Morgan [1989] implies global existence of solutions on fixed domains. We first consider the problem (2.1) with this general reaction term

$$(5.4) \quad f_i(\mathbf{u}) = \sum_{j=1}^m c_{ij} u_j + (-1)^i g(\mathbf{u}) + b_i,$$

with the following restrictions

$$(5.5) \quad c_{ij} \geq 0, \quad \text{for } i \neq j.$$

$$(5.6) \quad b_i \geq 0, \quad i = 1, \dots, m.$$

$$(5.7) \quad g(\mathbf{u})|_{u_i=0} = 0, \quad i = 1, \dots, m.$$

$$(5.8) \quad g(\mathbf{u}) \leq \left( \sum_{i=1}^m u_i \right)^p, \quad \text{for all } \mathbf{u} \in \mathbb{R}_+^m.$$

$$(5.9) \quad g \in C^1(\mathbb{R}_+^m; \mathbb{R}).$$

The motivation of this type of kinetics, as discussed by Murray [2003], is their role in the theory of biological oscillators due to a feedback mechanism.

**5.3. Proposition** (Lyapunov function) We show that problem (2.1) equipped with kinetics (5.4) is well posed, with Lyapunov function  $H(\mathbf{z}) := \sum_{i=1}^m z_i$ .

**Proof** Recalling that the initial data is bounded and nonnegative (2.2), we show that Assumption 3.4 is fulfilled, which implies  $\mathbb{R}_m^+$  is an invariant region for the solutions. Indeed, from (5.7) we have

$$(5.10) \quad f_i(\mathbf{u})|_{u_i=0} = \sum_{j=1}^m c_{ij} u_j + b_i = \sum_{j=1}^{i-1} c_{ij} u_j + \sum_{j=i+1}^m c_{ij} u_j + b_i.$$



Conditions (5.5) and (5.6) imply

$$(5.11) \quad f_i(\mathbf{u})|_{u_i=0} \geq b_i \geq 0 \quad \text{for all } \mathbf{u} \in \mathbb{R}_+^m.$$

Thus Assumption 3.4 is fulfilled due to (5.9). Now we show conditions (3.8)—(3.13) are fulfilled with  $r = 1$ . Clearly conditions (3.8)—(3.10) hold. Condition (3.11) holds with

$$a_{ij} := \begin{cases} 1 & \text{if } j = i, \\ 1 & \text{if } j = 1 \text{ and } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

$k_1 = \max_{i,j}(c_{ij})$ ,  $k_2 = \sum_{i=1}^m b_i$  and  $r = 1$ . Condition (3.12) holds since

$$(5.12) \quad h'_i f_i = f_i \leq \max_{i,j}(c_{ij}) \sum_{i=1}^m u_i + \max_i(b_i) + g(\mathbf{u}).$$

Using (5.8) we have

$$(5.13) \quad \begin{aligned} h'_i f_i &\leq \max_{i,j}(c_{ij}) \sum_{i=1}^m u_i + \max_i(b_i) + \left(\sum_{i=1}^m u_i\right)^p \\ &\leq \max_{i,j}(c_{ij}) H(u) + H(u)^p + \max_i(b_i) \leq kH(u)^q + b, \end{aligned}$$

where  $k, q, b \in \mathbb{R}_+$  represent constants that depend on the value of  $p$  in (5.8). Condition (3.13) holds since

$$(5.14) \quad \nabla H \cdot \mathbf{f} = \sum_{i=1}^m f_i \leq \max_{i,j}(c_{ij}) \sum_{i=1}^m u_i + \sum_{i=1}^m b_i \leq kH(u) + b,$$

where  $k = \max_{i,j}(c_{ij})$  and  $b = \sum_{i=1}^m b_i$ . A straightforward application of Theorem 4.5 thus completes the proof.  $\square$

**5.4. Examples.** The generic problem for which we showed global existence of solutions actually encompasses some of the more widely studied models in the theory of pattern formation such as the Gray-Scott model and the Brusselator. Below we present two examples of two species reaction terms for which our analysis implies global existence of solutions, the first of which is a restriction of the reaction term above.

- *Activator-depleted* substrate model: We consider the *activator-depleted* substrate model [Gierer and Meinhardt, 1972; Lefever and Prigogine, 1968; Schnakenberg, 1979] also known as the Brusselator model:

$$(5.15) \quad \begin{cases} f_1(u_1, u_2) &= \gamma(a - u_2^2 u_1), \\ f_2(u_1, u_2) &= \gamma(b - u_2 + u_2^2 u_1), \end{cases}$$

where  $0 < a, b, \gamma < \infty$ . The assumption of nonnegative initial data implies that the positive quadrant is invariant for our problem due to the fact that  $a, b > 0$ . If we take  $H(\mathbf{u}) = u_1 + u_2$ , then conditions (3.8)—(3.13) are fulfilled with  $r = 0$  which implies global existence of solutions on evolving domains via Theorem 4.5. The remaining constants for which conditions (3.8)—(3.13) hold are given in Table 1.

- Thomas reaction kinetics: The following model, proposed and studied experimentally by Thomas [1975], is based on a specific reaction involving the substrates oxygen and uric

acid which react in the presence of the enzyme uricase:

$$(5.16) \quad \begin{cases} f_1(u_1, u_2) &= \gamma(a - u_1 - g(u_1, u_2)), \\ f_2(u_1, u_2) &= \gamma(b - \alpha u_2 - g(u_1, u_2)), \end{cases}$$

where

$$(5.17) \quad g(u_1, u_2) = \frac{\kappa u_1 u_2}{1 + u_1 + \beta u_1^2},$$

and  $0 < \gamma, a, \alpha, b, \kappa, \beta < \infty$ . Once again the assumption of nonnegative initial data implies that the positive quadrant is invariant for our problem due to the fact that  $a, b > 0$ . If we again take  $H(\mathbf{u}) = u_1 + u_2$ , then conditions (3.8)—(3.13) are fulfilled with  $r = 0$  which implies global existence of solutions on evolving domains via Theorem 4.5.

The remaining constants for which conditions (3.8)—(3.13) hold are given in Table 1.

**5.5. Remark** (Invariant rectangles for the Thomas model) It can be shown, utilising the techniques of Smoller [1994], that there exist bounded invariant rectangles for the solutions of the Thomas model defined in (5.16). This implies global existence of solutions via Theorem 3.1. However the authors can show that it is possible to construct growth functions that fulfil Assumption 2.3, for which the bounded invariant rectangle can be made arbitrarily large. This necessitates the Lyapunov function approach to show existence and uniqueness of solutions.

Parameters	<i>Activator-depleted</i> substrate model	Thomas model
$a_{11}$	1	1
$a_{12}$	0	0
$a_{21}$	1	0
$a_{22}$	1	1
$k_1$	0	0
$k_2$	$\gamma(a + b)$	$\gamma(a + b)$
$k_3$	$\frac{\gamma}{2}$	0
$q$	3	0
$k_4$	$\gamma(\max(a, b))$	$\gamma(\max(a, b))$
$k_5$	0	0
$k_6$	$\gamma(a + b)$	$\gamma(a + b)$

TABLE 1. Terms from §3 for which conditions (3.8)—(3.13) hold for the kinetics defined in (5.15) and (5.16) respectively.

**5.6. Remark** (Further applications) The analysis can be applied to a large number of problems unrelated to the theory of pattern formation. Garvie and Trenchea [2009] provide an example applicable to ecology and the aforementioned paper of Morgan [1989] contains further examples as well as the numerous citations of said paper that use the approach on various problems.

## 6. NUMERICAL EXPERIMENTS

In this section we present numerical results on two-dimensional evolving domains to back-up the theoretical results of the previous sections. The numerical simulation of RDS's on growing domains is an extensive research area. Crampin et al. [2002] and Madzvamuse and Maini [2007] study mode doubling and tripling behaviour of the *activator-depleted* substrate model on one- and two-dimensional growing domains. The theoretical results derived above apply for any evolution

that fulfils Assumption 2.3. We present numerical results on a periodically evolving domain that exhibit spot splitting as well as spot annihilation and merging, which to the authors knowledge is as yet an unstudied area.

**6.1. Domain evolution.** We consider periodic domain evolution defined by

$$(6.1) \quad \rho(t) = 1 + 9 \sin \left( \frac{\pi t}{T} \right), \quad t \in [0, 1000 = T],$$

with the initial domain defined as  $\Omega_0 = [-0.25, 0.25]^2$  which grows to  $\Omega_{500} = [-2.5, 2.5]^2$  before contracting back to the original domain.

**6.2. Continuous problems.** We present results for the aforementioned *activator-depleted* model considered on a periodically evolving domain and its equivalent transformed system on a fixed domain as in §2.2. For example, on a periodically evolving domain the problem is stated as follows:

$$(6.2) \quad \begin{cases} \partial_t u_1(\mathbf{x}, t) + [\nabla \cdot (\mathbf{a} u_1)](\mathbf{x}, t) - \Delta u_1(\mathbf{x}, t) = 0.1 - [u_2^2 u_1](\mathbf{x}, t), & \text{for } \mathbf{x} \in \Omega_t \\ \partial_t u_2(\mathbf{x}, t) + [\nabla \cdot (\mathbf{a} u_2)](\mathbf{x}, t) - 0.01 \Delta u_2(\mathbf{x}, t) = 0.9 - u_2(\mathbf{x}, t) + [u_2^2 u_1](\mathbf{x}, t), & \text{and } t \in (0, T], \\ [\vec{\nu} \cdot \nabla \mathbf{u}](\mathbf{x}, t) = 0, & \mathbf{x} \in \partial \Omega_t, t > 0, \end{cases}$$

where  $\mathbf{x} = \rho(t)\boldsymbol{\xi}$  and  $a_i(\mathbf{x}, t) = \dot{\rho}(t)\boldsymbol{\xi}$ ,  $i = 1, 2$ . Equivalently, the following transformed equations are obtained on a fixed domain,

$$(6.3) \quad \begin{cases} \partial_t u_1 + 2 \frac{\dot{\rho}}{\rho} u_1 - \frac{1}{\rho^2} \Delta u_1 = 0.1 - u_2^2 u_1, \\ \partial_t u_2 + 2 \frac{\dot{\rho}}{\rho} u_2 - \frac{0.01}{\rho^2} \Delta u_2 = 0.9 - u_2 + u_2^2 u_1, \\ [\vec{\nu} \cdot \nabla \mathbf{u}](\boldsymbol{\xi}, t) = 0, \end{cases} \quad \begin{array}{l} \text{on } \Omega_0 \times (0, T], \\ \boldsymbol{\xi} \in \partial \Omega_0, t > 0. \end{array}$$

In both cases we take identical initial conditions as small perturbations around the homogenous steady state of (1.0, 0.9) obtained in the absence of domain evolution.

**6.3. Numerical schemes.** We employ a Galerkin finite element method for the spatial approximation and an implicit-explicit modified backward Euler scheme for the time integration. Discretising in time we divide the time interval  $[0, T]$  into a partition of  $N$  uniform subintervals,  $0 = t_0 < \dots < t_N = T$  and denote by  $\tau := t_n - t_{n-1}$  the time step. For the spatial discretisation we introduce a regular triangulation  $\mathcal{T}^0$  of  $\Omega_0$  with  $K \in \mathcal{T}^0$  an open simplex. We define the following shorthand for a function of time,  $f(t_n) =: f^n$ .

We define the finite element space on the initial domain  $\mathbb{V}^0 \subset H^1(\Omega_0)$  as,

$$(6.4) \quad \mathbb{V}^0 := \{\Phi \in H^1(\Omega_0) : \Phi|_K \in \mathbb{P}^1 \text{ for all } K \in \mathcal{T}^0\},$$

where  $\mathbb{P}^1$  denotes the space of polynomials no higher than degree 1. For the numerical simulation of equation (6.2) we require finite element spaces defined on the evolving domain. We construct the finite element spaces  $\mathbb{V}^n$  according to the following relation between the basis functions of  $\mathbb{V}^n$  and  $\mathbb{V}^0$ .

$$(6.5) \quad \Psi^n = \Psi(\rho^n \boldsymbol{\xi}, t_n) = \Phi(\boldsymbol{\xi}) \quad n = 1, \dots, N.$$

Thus the family of finite element spaces on the evolving domain  $\mathbb{V}^n \subset H^1(\Omega_{t_n})$   $n = 1, \dots, N$  may be defined as,

$$(6.6) \quad \mathbb{V}^n := \{\Psi^n \in H^1(\Omega_{t_n}) : \Psi^n|_K \in \mathbb{P}^1 \text{ for all } K \in \mathcal{T}^n\},$$

where we have used the fact that the domain evolution is linear with respect to space. We approximate the initial conditions in both schemes by

$$(6.7) \quad \mathbf{U}^0 = \mathcal{I}\mathbf{u}_0(\boldsymbol{\xi}) \quad \text{for all } \boldsymbol{\xi} \in \Omega_0,$$

where  $\mathcal{I}$  is the standard Lagrange interpolant. The finite element scheme to approximate the solution to equation (6.2) aims to find  $U_1^n, U_2^n \in \mathbb{V}^n, n = 1, \dots, N$  such that

$$(6.8) \quad \begin{cases} \frac{1}{\tau} \langle U_1^n, \Psi^n \rangle + \langle \nabla U_1^n, \nabla \Psi^n \rangle = \langle 0.1 - (U_2^{n-1})^2 U_1^n, \Psi^n \rangle + \frac{1}{\tau} \langle U_1^{n-1}, \Psi^{n-1} \rangle, \\ \frac{1}{\tau} \langle U_2^n, \Psi^n \rangle + 0.01 \langle \nabla U_2^n, \nabla \Psi^n \rangle = \langle 0.9 - U_2^n + U_2^{n-1} U_1^n U_2^n, \Psi^n \rangle + \frac{1}{\tau} \langle U_2^{n-1}, \Psi^{n-1} \rangle, \end{cases}$$

for all  $\Psi^n \in \mathbb{V}^n, n = 1, \dots, N$ . Similarly the finite element scheme to approximate the solution to equation (6.3) aims to find  $W_1^n, W_2^n \in \mathbb{V}^0, n = 1, \dots, N$  such that

$$(6.9) \quad \begin{cases} \frac{1}{\tau} \langle W_1^n - W_1^{n-1}, \Phi \rangle + \frac{1}{(\rho^n)^2} \langle \nabla W_1^n, \nabla \Phi \rangle + \frac{2\rho^n}{\rho^n} \langle W_1^n, \Phi \rangle = \langle 0.1 - (W_2^{n-1})^2 W_1^n, \Phi \rangle, \\ \frac{1}{\tau} \langle W_2^n - W_2^{n-1}, \Phi \rangle + \frac{0.01}{(\rho^n)^2} \langle \nabla W_2^n, \nabla \Phi \rangle + \frac{2\rho^n}{\rho^n} \langle W_2^n, \Phi \rangle = \langle 0.9 - W_2^n + W_2^{n-1} W_1^n W_2^n, \Phi \rangle, \end{cases}$$

for all  $\Phi \in \mathbb{V}^0$ .

We solved the models in C utilising the FEM library ALBERTA by Schmidt and Siebert [2005]. We used the conjugate gradient solver to compute our discrete solutions. We took an initial triangulation  $\mathcal{T}^0$  with 8321 nodes, a uniform mesh diameter of  $2^{-6}$  and a fixed timestep of  $10^{-3}$ . PARAVIEW was used to display our results.

**6.4. Results.** Figures 1 and 2 show snapshots of the activator profile corresponding to the *activator-depleted* system (5.15). The inhibitor profiles have been omitted as they are  $180^\circ$  out of phase to the activator profiles. We have verified numerically that there is very little difference between the discrete solution corresponding to system (6.8) mapped to the fixed domain and the discrete solution corresponding to system (6.9) defined on a fixed domain, as is expected from the results in §2.2.

The figures illustrate the mode doubling phenomena that occurs as the domain grows as well as the spot annihilation and spot merging phenomena that occurs as the domain contracts. In Figure 3 we present in more detail the novel spot merging phenomena observed on the contracting domain. It is still unclear whether the spot merging phenomenon is in fact a special case of the spot annihilation phenomenon, that occurs when the modes are of sufficient proximity to influence each other.

We note that the mode transition sequence, i.e., the number of spots, is different when the domain grows to when it contracts. The difference in the mechanism of mode transitions on growing and contracting domains is an area in which very little work has been done and these initial numerical results indicate the need for further exploration of this area.

## 7. CONCLUSION

Many problems in biology and biomedicine involve growth. In developmental biology recent advances in experimental data collection allow experimentalists to capture the emergence of pattern structure formation during growth development of the organism or species. Such experiments include the formation of spot patterns on the surface of the eel, patterns emerging on the surface of the Japanese flounder and butterfly wing patterns forming during the growth development of the imaginal wing disc. In all these examples, patterns form during growth development.

Since the seminal paper by Turing [1952] which considered linear models that could give rise to spatiotemporal solutions on fixed domains due the process of diffusion-driven instability, a lot of theoretical results on global existence of such solutions have been derived and proved for highly nonlinear mathematical models Rothe [1984]; Smoller [1994]. Only recently, mathematical models on growing domains have been derived from first principles in order to incorporate the effects of domain evolution into the models Crampin et al. [1999]; Madzvamuse [2000]. In all these studies, very little analysis has been done up to now to extend the theoretical global existence results to models defined on evolving domains.

Under suitable assumptions, we have extended existence results from problems posed on fixed domains to problems posed on an evolving domain. We have illustrated the applicability of the existence results of Morgan [1989] to problems on evolving domains. We have shown that global existence of solutions to many commonly encountered RDS's on fixed domains implies global existence of solutions to the same RDS's on a class of evolving domains. The results are significant in the theory of pattern formation especially in fields such as developmental biology where problems posed on evolving domains are commonly encountered. Our results hold with no assumptions on the sign of the growth rate, which may prove useful in other fields where monotonic domain growth is not valid from a modelling perspective. The applicability of our results is demonstrated by considering different forms of domain evolution (linear, logistic and exponential).

In order to validate our theoretical findings, we presented results on a periodically evolving domain. Our results illustrate the well-known period-doubling phenomenon during domain growth but more interesting and surprising is the development of spot annihilation and spot merging phenomena during contraction. This raises new questions about bifurcation analysis on growing and contracting domains.

One of our primary goals is the numerical analysis of finite element approximations of RDS's on evolving domains. The classical existence results obtained will be an important tool in future work. Numerical experiments have been carried out and they illustrate the need for further numerical analysis especially in the case of contracting domains. Extension of our work onto domains with more complex evolution is another area for future research.

## ACKNOWLEDGMENTS

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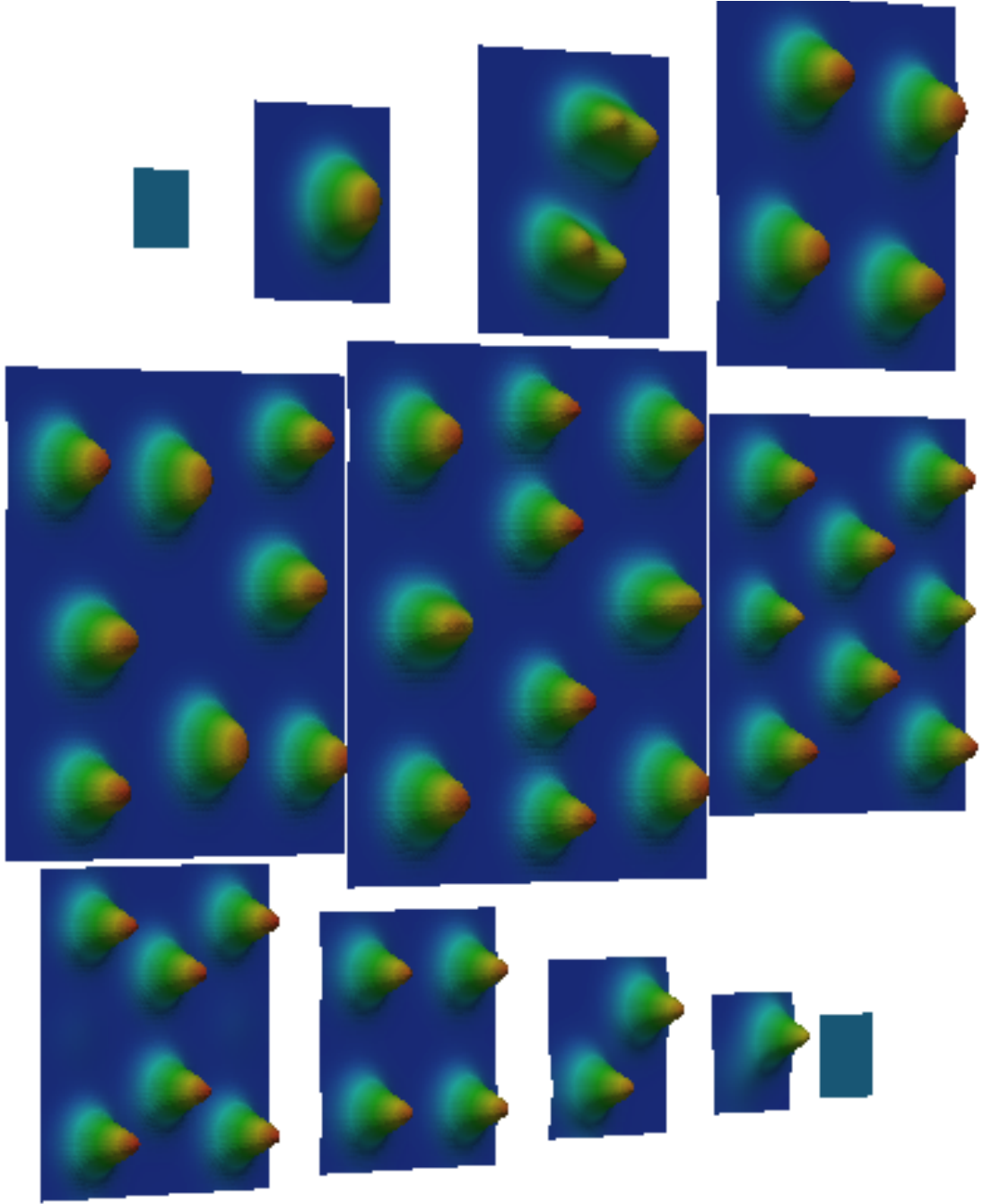


FIGURE 1. Snapshots of the discrete solution  $U_2$  corresponding to system (6.8) at times 0, 50, 160, 220, 380, 500, 700, 740, 820, 900, 980 and 1000 reading from left to right and then top to bottom. For parameter and numerical values see §6.3. The solution exhibits a mode doubling sequence of 1, 2, 4, 8 and finally 16 as the domain grows. As the domain contracts the spots are annihilated in a sequence of 16, 12, 8, 4, 2 and the final transition to a single spot occurs via merging, with the final domain exhibiting no patterns.

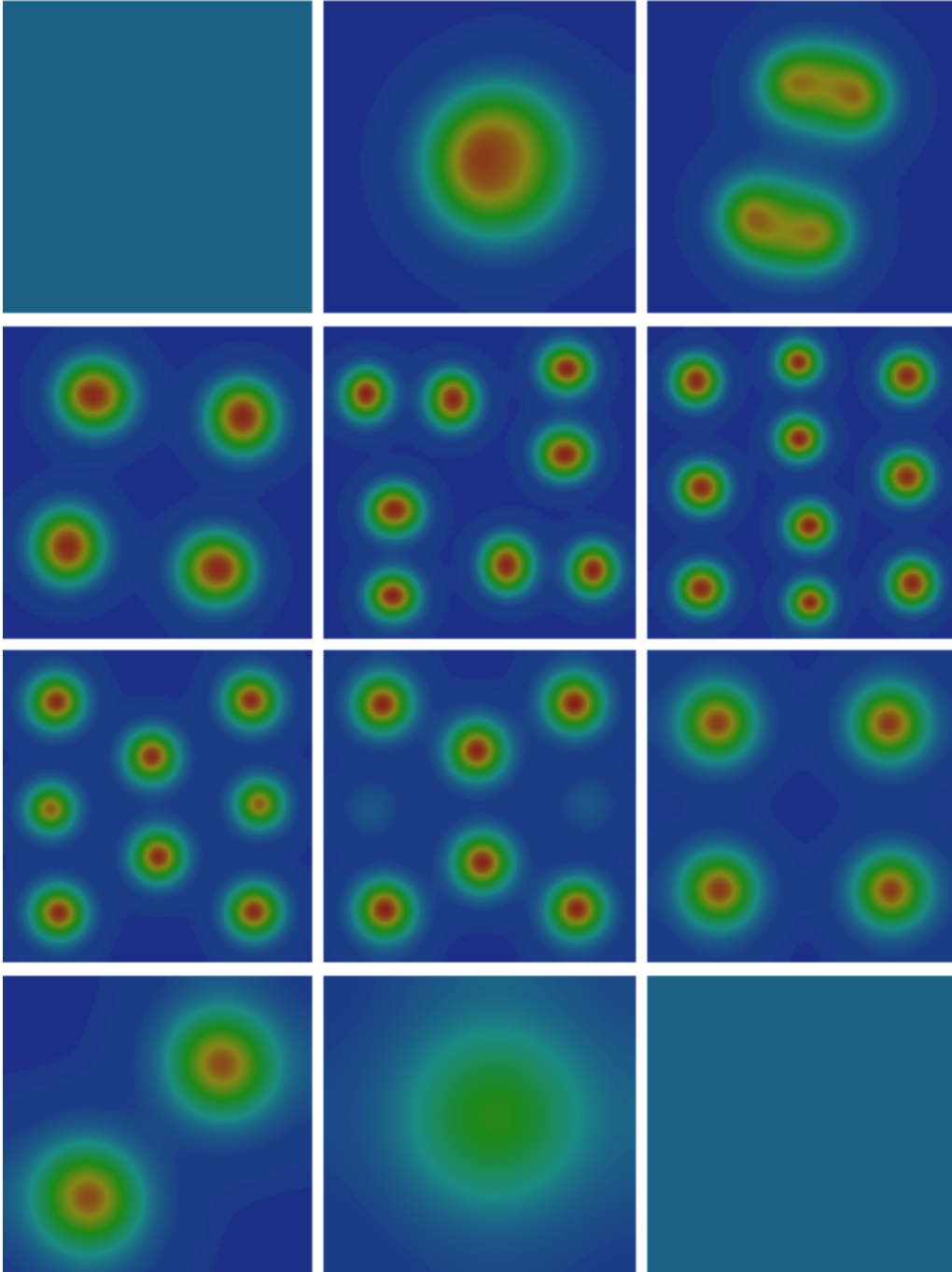


FIGURE 2. Snapshots of the discrete solution  $W_2$  corresponding to system (6.9) at times 0, 50, 160, 220, 380, 500, 700, 740, 820, 900, 980 and 1000 reading from left to right and then top to bottom. For parameter and numerical values see §6.3. The mode transition follows exactly that of Figure 1, corroborating the results in §2.2.

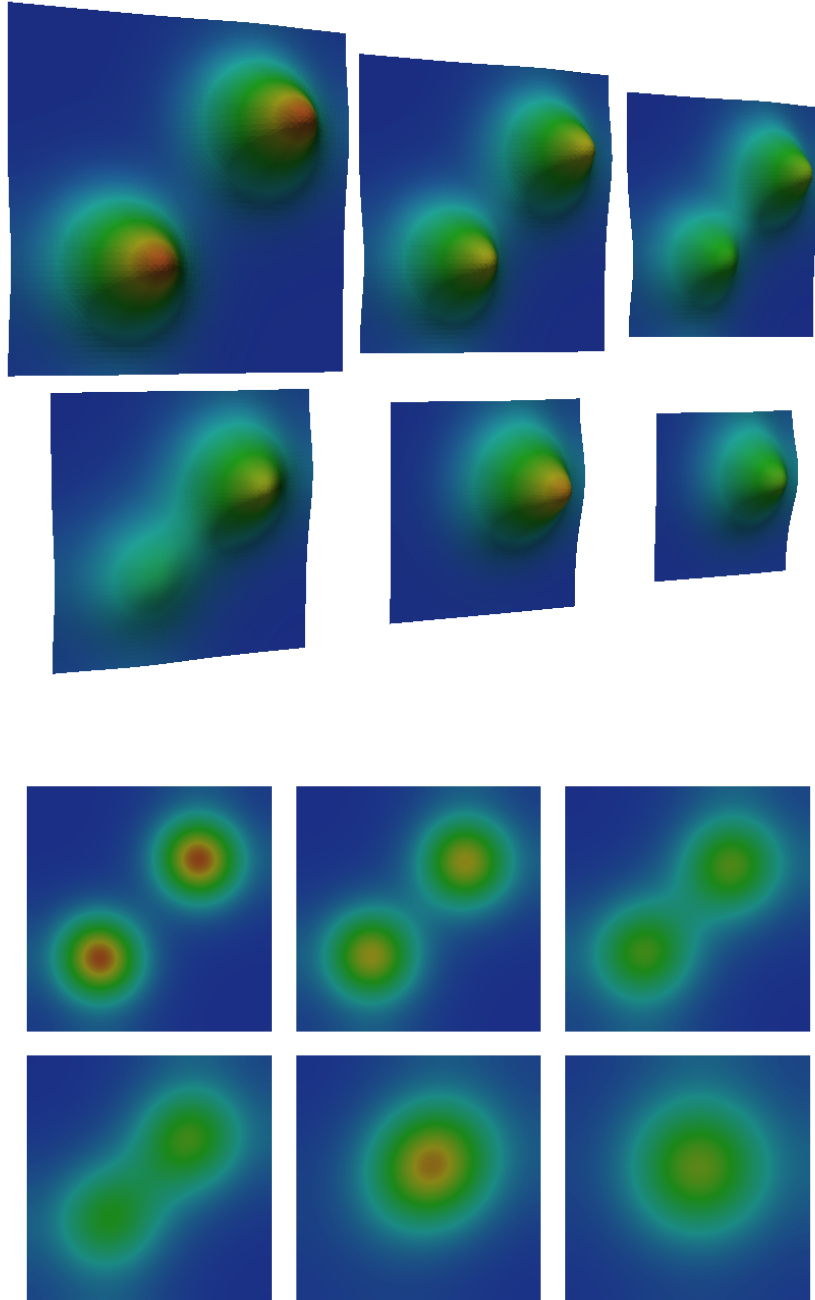


FIGURE 3. Snapshots of the discrete solution corresponding to system (6.8) (top) and system (6.9) (bottom) at times 930, 940, 950, 955, 965, 975. The spot merging phenomena observed in the transition from two spots to one spot is displayed.



APPENDIX A. A PRIORI ESTIMATES FOR SYSTEMS WITH TIME DEPENDENT  $f$ 

In this section we prove Theorem 3.7 remains applicable for solutions of Problem (2.8). For conciseness we focus on assertion (3.14). We adapt the proof of Morgan [1989, Th. 3.2] to our purposes. In Appendix B, we use the a priori estimates obtained in this section to prove the global existence of classical solutions to Problem (2.8). We state the main result of this section in the following Lemma.

**A.1. Lemma** (An a priori estimate for solutions of Problem (2.8)) Suppose Assumptions 2.3, 3.4 and 4.1 hold and let (3.8)—(3.10) and (3.13) hold. Then, assertion (3.14) is valid with  $\tilde{\mathbf{u}}$  a solution of Problem (2.8).

**Proof** Let  $\tilde{\mathbf{f}}$  be as defined in (4.2). From the proof of Lemma 4.1, (4.7) holds. We split the remainder of the proof into steps.

Step 1: We first show the following inequality for  $H$ :

$$(A.1) \quad \begin{aligned} H(\tilde{\mathbf{u}}(\vec{\xi}, t)) &\leq \int_{\tau}^t \sum_{i=1}^m D_i \Delta_{\vec{\xi}} h_i(\tilde{u}_i(\vec{\xi}, r)) + (k_5 C_1^2 + k_{12}) H(\tilde{\mathbf{u}}(\vec{\xi}, r)) \, dr \\ &\quad + H(\tilde{\mathbf{u}}(\vec{\xi}, \tau)) + k_6 C_1^2(t - \tau) \text{ for } \vec{\xi} \in \Omega_0 \text{ and } t < T_{\max}. \end{aligned}$$

From (2.8) we have for  $(\vec{\xi}, t) \in \Omega_0 \times (0, T_{\max})$ ,

$$(A.2) \quad \nabla H(\tilde{\mathbf{u}}(\vec{\xi}, t)) \cdot \partial_t \tilde{\mathbf{u}}(\vec{\xi}, t) = \nabla H(\tilde{\mathbf{u}}(\vec{\xi}, t)) \cdot (D \Delta \tilde{\mathbf{u}}(\vec{\xi}, t) + \tilde{\mathbf{f}}(\tilde{\mathbf{u}}(\vec{\xi}, t), t)).$$

Using (3.8), (4.7) and (A.2) we obtain the following generalisation of Morgan [1989, (3.1)],

$$(A.3) \quad \nabla H(\tilde{\mathbf{u}}(\vec{\xi}, t)) \cdot \partial_t \tilde{\mathbf{u}}(\vec{\xi}, t) \leq \sum_{i=1}^m D_i \Delta_{\vec{\xi}} h_i(\tilde{u}_i(\vec{\xi}, t)) + (k_5 C_1^2 + k_{12}) H(\tilde{\mathbf{u}}(\vec{\xi}, t)) + k_6 C_1^2,$$

where we have used the convexity of  $H$  (3.9). Integrating (A.3) in time gives (A.1).

Step 2: We use (A.1) to construct an appropriate barrier function with a view to applying the maximum principle, corresponding to Morgan [1989, (3.2)—(3.4)]. Introducing an arbitrary  $T^* < T_{\max}$ , we define

$$(A.4) \quad w(\vec{\xi}, t) := \int_{\tau}^t \sum_{i=1}^m \frac{D_i}{D^*} h_i(\tilde{u}_i(\vec{\xi}, r)) \, dr \text{ for } \vec{\xi} \in \Omega_0 \text{ and } 0 \leq \tau < t \leq T^*,$$

where  $D^* := \max_i(D_i)$ . Observe that from (3.8) and (A.4), we have for  $0 \leq \tau < t \leq T^*$

$$(A.5) \quad \begin{aligned} \left\| \int_{\tau}^t H(\tilde{\mathbf{u}}(\cdot, s)) \, ds \right\|_{L_{\infty}(\Omega_0)} &= \left\| \int_{\tau}^t \sum_{i=1}^m h_i(\tilde{u}_i(\cdot, s)) \, ds \right\|_{L_{\infty}(\Omega_0)} \\ &\leq \max_i \left( \frac{D^*}{D_i} \right) \|w(\cdot, s) \, ds\|_{L_{\infty}(\Omega_0)}. \end{aligned}$$

From (A.1) and (A.4) we obtain

$$(A.6) \quad \begin{cases} \partial_t w(\vec{\xi}, t) \leq D^* \Delta w(\vec{\xi}, t) + L + M w(\vec{\xi}, t) + k_6 C_1^2(t - \tau), & \vec{\xi} \in \Omega_0, 0 \leq \tau < t \leq T^* \\ [\nu \cdot \nabla w](\vec{\xi}, t) = 0, & \vec{\xi} \in \partial\Omega_0, t \in (\tau, T^*] \\ w(\vec{\xi}, \tau) = 0, & \vec{\xi} \in \Omega_0, \end{cases}$$

where

$$(A.7) \quad L := \|H(\tilde{\mathbf{u}}(\cdot, \tau))\|_{L_\infty(\Omega_0)} \text{ and } M := \frac{(k_{12} + C_1^2 k_5)D^*}{\min_i(D_i)}.$$

Note we have used (2.8) to obtain the boundary conditions. For the purposes of applying the maximum principle we define a barrier function

$$(A.8) \quad \hat{w}(\boldsymbol{\xi}, t) := w(\boldsymbol{\xi}, t) - \frac{L + K_6 C_1^2 T^*}{M} \text{ for } \vec{\xi} \in \Omega_0 \text{ and } 0 \leq \tau < t \leq T^*.$$

From (A.6) and (A.4) we have

$$(A.9) \quad \begin{cases} \partial_t \hat{w}(\boldsymbol{\xi}, t) \leq D^* \Delta \hat{w}(\boldsymbol{\xi}, t) + M \hat{w}(\boldsymbol{\xi}, t) & \vec{\xi} \in \Omega_0, 0 \leq \tau < t \leq T^* \\ [\boldsymbol{\nu} \cdot \nabla \hat{w}](\boldsymbol{\xi}, t) = 0, & \boldsymbol{\xi} \in \partial\Omega_0, t \in (\tau, T^*] \\ \hat{w}(\boldsymbol{\xi}, \tau) \leq 0, & \boldsymbol{\xi} \in \Omega_0. \end{cases}$$

Step 3: We use the maximum principle to complete the proof. Applying the strong maximum principle for parabolic problems [Sperb, 1981, Th. 2.9, Rem. (a) pg. 21] to (A.9) and noting the positivity of  $w$ , we have

$$(A.10) \quad -\frac{L + K_6 C_1^2 T^*}{M} \leq \hat{w}(\boldsymbol{\xi}, t) \leq 0 \text{ for all } \vec{\xi} \in \Omega_0 \text{ and for } t \in (\tau, T^*].$$

From (A.8) and (A.10) we have

$$(A.11) \quad 0 \leq \hat{w}(\boldsymbol{\xi}, t) \leq \frac{L + K_6 C_1^2 T^*}{M} \text{ for all } \vec{\xi} \in \Omega_0 \text{ and for } t \in (\tau, T^*].$$

We conclude from (A.11) that

$$(A.12) \quad \|w(\cdot, t)\|_{L_\infty(\Omega_0)} \leq \frac{L + K_6 C_1^2 T^*}{M} \text{ for } t \in (\tau, T^*].$$

Since  $T^*$  was arbitrary, combining (A.5) and (A.12) completes the proof of the Lemma.  $\square$

For completeness, we sketch the proof of assertion (3.15) with  $\tilde{\mathbf{u}}$  a solution of Problem (2.8). In (A.3) we denote  $K_7 := k_5 C_1^2 + k_{12}$  and  $K_8 := k_6 C_1^2$ , where  $K_7, K_8$  correspond to the terms on the right hand side of Morgan [1989, (3.1)]. Assertion (3.15) follows from the proofs of Morgan [1989, Th. 3.3 and 3.4].

## APPENDIX B. GLOBAL EXISTENCE RESULTS FOR SYSTEMS WITH TIME DEPENDENT $f$

The main result of this section is Theorem B.4, a special case of Theorem 3.8. It is applicable to solutions of Problem (2.8). Theorem B.4 is enough for our purposes as seen in the examples in §5. To prove the Theorem, we will modify the proof of Morgan [1989, Th. 2.2] with stronger control of the parameter  $r$ , that appears in (3.11).

We start with two Lemmas from Morgan [1989, Lem. 4.1, Lem. 4.2, (4.12)] which follow from the results of Ladyzhenskaya et al. [1968][Th. 9.1 p.341]. We then use a duality approach to prove global existence of classical solutions to (2.8).

**B.1. Lemma** (Global existence) Let  $\tilde{\mathbf{u}}$  be the solution of Problem (2.8). Let the function  $H$  fulfil conditions (3.8)—(3.10) in §3.6 and let the polynomial growth restriction on  $\tilde{\mathbf{f}}$  (4.6) hold. Let  $T_{\max}$  be as defined in (3.1) and suppose that,

$$(B.1) \quad \begin{cases} \text{for } 0 \leq \tau < T < T_{\max} \text{ and for all } p \in (1, \dots, \infty) \\ \text{there exist } M_p, N_p > 0 \text{ and } 0 < \delta_p < 1 \text{ such that} \\ \sum_{i=1}^m \|h_i(\tilde{u}_i)\|_{L_p(\Omega_0 \times (\tau, T))} \leq M_p(T - \tau) + N_p(T - \tau) \|H(\tilde{\mathbf{u}})\|_{L_p(\Omega_0 \times (\tau, T))}^{\delta_p}, \end{cases}$$

then  $T_{\max} = \infty$ .

**B.2. Definition** (Dual problem) A key ingredient of the proof of Theorem B.4 is the dual solution  $\psi$ . Where for  $i = 1, \dots, m$ ,  $\psi$  is the solution of the scalar equation

$$(B.2) \quad \begin{cases} \partial_t \psi(\xi, t) = -D_i \Delta \psi(\xi, t) - \theta(\xi, t) & \text{for } \xi \in \Omega_0 \text{ and } 0 \leq t < T < T_{\max} \\ [\nu \cdot \nabla \psi](\xi, t) = 0, & \xi \in \partial\Omega_0, t \in [0, T) \\ \psi(\xi, T) = 0, & \xi \in \Omega_0, \end{cases}$$

where  $\theta \geq 0$  is such that, for all  $p \in (1, \dots, \infty)$ ,  $\|\theta\|_{L_p(\Omega_0 \times [0, T])} = 1$ .

**B.3. Lemma** (Control of the solution to the dual problem (B.2)) Let  $\psi$  be as defined in B.2. For  $i = 1, \dots, m$  and for  $p \in (1, \dots, \infty)$ , there exists  $C_{p,T} > 0$  such that,

$$(B.3) \quad \|\psi\|_{L_p(\Omega_0; L_\infty[0, T])} \leq C_{p,T}$$

We now state the main result of this section. Namely, the applicability of a special case of Theorem 3.8 to solutions of Problem (2.8).

**B.4. Theorem** (Sufficient conditions for global existence of solutions to Problem (2.8)) Let Assumptions 2.3, 3.4 and 4.1 hold. Let  $H$ ,  $\mathbf{f}$  and  $r$  satisfy the conditions in §3.6 with  $r \leq 1$  (cf. (3.11)), i.e., Problem (3.4) admits a global classical solution by Theorem 3.8. Then, Problem (2.8) admits a global classical solution, i.e.,  $T_{\max} = \infty$  (cf. (3.1)).

**Proof** We proceed by contradiction. Assume  $T_{\max} < \infty$ . Let  $\tilde{\mathbf{u}}$  and  $\tilde{\mathbf{f}}$  (cf. 4.2) be the solution and zero order term of Problem (2.8) respectively. From the proof of Lemma 4.3 the polynomial growth restriction (4.6) is satisfied by  $\tilde{\mathbf{f}}$  and  $H$ . Since  $T_{\max} < \infty$ , Lemma B.1 implies that (B.1)

does not hold. Let  $j \in [1, \dots, m]$  denote the smallest  $k$  for which  $\sum_{i=1}^k \|h_i(\tilde{u}_i)\|_{L_p(\Omega_0 \times (\tau, T))}$  does not satisfy (B.1). From the proof of Lemma 4.3, the intermediate sum condition (4.5) is satisfied for with  $\tilde{r} = 1$  (cf. (4.5)). From Lemma A.1, we have the a priori estimate (3.14).

We will show (3.14) and (4.5) imply (B.1) is satisfied for  $j$ , obtaining a contradiction. We split the remainder of the proof into steps.

Step 1: We first show the following inequality (corresponding to Morgan [1989, (4.6)–(4.9)]):  
For  $0 < T < T_{\max}$

$$\begin{aligned}
 (B.4) \quad & \int_{\Omega_0} \int_0^T \sum_{i=1}^j a_{ji} \frac{D_i}{D_j} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \\
 & \leq \int_0^T \int_{\Omega_0} \sum_{i=1}^{j-1} a_{ji} \left(1 - \frac{D_i}{D_j}\right) h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \partial_s \psi(\boldsymbol{\xi}, s) \\
 & \quad + \psi(\boldsymbol{\xi}, s) \left( (k_1 C_1^2 + k_7) H(\tilde{\mathbf{u}}(\boldsymbol{\xi}, s)) + k_8 \right) \, d\boldsymbol{\xi} \, ds \\
 & \quad + \sum_{i=1}^j a_{ji} \psi(\boldsymbol{\xi}, 0) h_i(\tilde{u}_i^0(\boldsymbol{\xi})) \, d\boldsymbol{\xi} \\
 & := I_1 + I_2 + I_3.
 \end{aligned}$$

From (B.2) we have for  $i \in [1, \dots, m]$

$$\begin{aligned}
 (B.5) \quad & \int_0^T \int_{\Omega_0} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \\
 & = \int_0^T \int_{\Omega_0} -h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) [\partial_s \psi + D_j \Delta \psi](\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \\
 & \leq \int_0^T \int_{\Omega_0} -h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \partial_s \psi(\boldsymbol{\xi}, s) \\
 & \quad - \frac{D_j}{D_i} \psi(\boldsymbol{\xi}, s) D_i \Delta_{\boldsymbol{\xi}} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \, d\boldsymbol{\xi} \, ds,
 \end{aligned}$$

where we have used integration by parts and the homogenous Neumann boundary conditions. From Problem (2.8) and the convexity of  $H$  (3.9), we have

$$\begin{aligned}
 (B.6) \quad & - \int_0^T \int_{\Omega_0} \psi(\boldsymbol{\xi}, s) D_i \Delta_{\boldsymbol{\xi}} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \, d\boldsymbol{\xi} \, ds \\
 & \leq \int_0^T \int_{\Omega_0} \psi(\boldsymbol{\xi}, s) h'_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \left( \tilde{f}_i(\tilde{\mathbf{u}}(\boldsymbol{\xi}, s), s) - \partial_s \tilde{u}_i(\boldsymbol{\xi}, s) \right) \, d\boldsymbol{\xi} \, ds \\
 & = \int_0^T \int_{\Omega_0} \psi(\boldsymbol{\xi}, s) h'_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \tilde{f}_i(\tilde{\mathbf{u}}(\boldsymbol{\xi}, s), s) \\
 & \quad + h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \partial_s \psi(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds + \int_{\Omega_0} \psi(\boldsymbol{\xi}, 0) h_i(\tilde{u}_i^0(\boldsymbol{\xi})) \, d\boldsymbol{\xi},
 \end{aligned}$$

where we have used integration by parts and the final condition of (B.2). Combining (B.6) and (B.5), we obtain

$$\begin{aligned}
 (B.7) \quad & \int_0^T \int_{\Omega_0} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \leq \int_{\Omega_0} \int_0^T \left( \frac{D_j}{D_i} - 1 \right) h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \partial_s \psi(\boldsymbol{\xi}, s) \\
 & \quad + \frac{D_j}{D_i} \psi(\boldsymbol{\xi}, s) h'_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \tilde{f}_i(\tilde{\mathbf{u}}(\boldsymbol{\xi}, s), s) \, d\boldsymbol{\xi} \, ds \\
 & \quad + \frac{D_j}{D_i} \psi(\boldsymbol{\xi}, 0) h_i(\tilde{u}_i^0(\boldsymbol{\xi})) \, d\boldsymbol{\xi}.
 \end{aligned}$$

Summing (B.7) over  $i \leq j$  and using condition (4.5) with  $\tilde{r} = 1$ , we obtain (B.4). For the case  $j = 1$ , we have introduced the convention  $\sum_{i=1}^0 (\cdot) = 0$ .

Step 2: We shall use Lemma B.3 and (B.4) to obtain the following inequality (as in Morgan [1989, (4.10)–(4.16)]): For all  $p \in (1, \dots, \infty)$  there exists  $K_{p,T} > 0$  and  $0 < \delta_p < 1$  independent of  $\tilde{\mathbf{u}}$  and  $\theta$  such that,

$$(B.8) \quad \int_0^T \int_{\Omega_0} \sum_{i=1}^j a_{ji} \frac{D_i}{D_j} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) d\boldsymbol{\xi} ds \leq K_{p,T} \left( 1 + \|H(\tilde{\mathbf{u}})\|_{L_p(\Omega_0 \times [0,T])}^{\delta_p} \right).$$

Let  $p, q \in (1, \dots, \infty)$  be such that,  $\frac{1}{p} + \frac{1}{q} = 1$ . Dealing firstly with  $I_1$  (cf. (B.4)), we have by Hölder's inequality and the regularity estimate (B.3)

$$(B.9) \quad \begin{aligned} I_1 &\leq \sum_{i=1}^{j-1} a_{ji} \left| 1 - \frac{D_i}{D_j} \right| C_{q,T} T \|h_i(\tilde{u}_i)\|_{L_p(\Omega_0 \times [0,T])} \\ &\leq \sum_{i=1}^{j-1} a_{ji} \left| 1 - \frac{D_i}{D_j} \right| C_{q,T} T \left( M_p + N_p \|H(\tilde{\mathbf{u}})\|_{L_p(\Omega_0 \times [0,T])}^{\delta_q} \right), \end{aligned}$$

where we have the assumption that (B.1) is valid for  $i < j$ . Dealing with  $I_2$ , we have by Hölder's inequality and the regularity estimate (B.3)

$$(B.10) \quad \begin{aligned} I_2 &\leq \int_{\Omega_0} \|\psi(\boldsymbol{\xi}, \cdot)\|_{L_\infty[0,T]} \left( \int_0^T ((k_1 C_1^2 + k_7) H(\tilde{\mathbf{u}}(\boldsymbol{\xi}, s)) + k_8 ds) d\boldsymbol{\xi} \right. \\ &\leq C_{q,T} \left( (k_1 C_1^2 + k_7) \left\| \int_0^T H(\tilde{\mathbf{u}}(\cdot, s)) \right\|_{L_q(\Omega_0)} + k_8 T \right) \\ &\leq C_{q,T} \left( g(T) |\Omega_0|^{1/p} + k_8 T \right), \end{aligned}$$

for some  $g \in C[0, \infty)$ . Where we have used the a priori estimate (3.14). Finally dealing with  $I_3$  using Hölder's inequality and estimate (B.3) we have

$$(B.11) \quad \begin{aligned} I_3 &\leq \sum_{i=1}^j a_{ji} C_{q,T} \left\| h_i(\tilde{u}_i^0) \right\|_{L_p(\Omega_0 \times [0,T])} \\ &\leq \sum_{i=1}^j a_{ji} C_{q,T} C_p, \end{aligned}$$

where we have used the boundedness of  $\tilde{\mathbf{u}}^0$  and condition (3.9). Combining (B.9), (B.10), (B.11) and (B.4) yields (B.8).

Step 3: We now show (B.1) holds for  $j$ . Let  $p, q \in (1, \dots, \infty)$ , be such that,  $\frac{1}{p} + \frac{1}{q} = 1$ . We recall, from (3.9) and Definition B.2, that for  $i \in [1, \dots, m]$ ,  $h_i, \theta \geq 0$  and  $\|\theta\|_{L_q(\Omega_0 \times (0,T))} = 1$ .

Using duality we obtain

$$\begin{aligned}
 (B.12) \quad & \min_{i \leq j} \left( a_{ji} \frac{D_i}{D_j} \right) \sum_{i=1}^j \|h_i(\tilde{u}_i)\|_{L_p(\Omega_0 \times (0, T))} \\
 &= \min_{i \leq j} \left( a_{ji} \frac{D_i}{D_j} \right) \int_0^T \int_{\Omega_0} \sum_{i=1}^j h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \\
 &\leq \int_0^T \int_{\Omega_0} \sum_{i=1}^j a_{ji} \frac{D_i}{D_j} h_i(\tilde{u}_i(\boldsymbol{\xi}, s)) \theta(\boldsymbol{\xi}, s) \, d\boldsymbol{\xi} \, ds \\
 &\leq K_{p,T} \left( 1 + \|H(\tilde{\mathbf{u}})\|_{L_p(\Omega_0 \times [0, T])}^{\delta_p} \right),
 \end{aligned}$$

where we have used (B.8).

Thus we have a contradiction and we conclude  $T_{\max} = \infty$  completing the proof.  $\square$

The proof of Theorem 3.8 follows from more technical use of Hölder's inequality in (B.10) and in the case of assertion (3.17) we also require the a priori estimate (3.15). We refer to Morgan [1989, (4.13)–(4.16) and (4.27)–(4.19)] for specific details.

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